Symmetric path integrals for stochastic equations with multiplicative noise

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A Langevin equation with multiplicative noise is an equation schematically of the form $d\mathbf{q}/dt=-\mathbf{F}(\mathbf{q})+e(\mathbf{q})\boldsymbol{\xi}$, where $e(\mathbf{q})\boldsymbol{\xi}$ is Gaussian white noise whose amplitude $e(\mathbf{q})$ depends on $\mathbf{q}$ itself. I show how to convert such equations into path integrals. The definition of the path integral depends crucially on the convention used for discretizing time, and I specifically derive the correct path integral when the convention used is the natural, time-symmetric one whose time derivatives are $(\mathbf{q}_t-\mathbf{q}_{t-\Delta t})/\Delta t$ and coordinates are $(\mathbf{q}_t+\mathbf{q}_{t-\Delta t})/2$. (This is the convention that permits standard manipulations of calculus on the action, like naive integration by parts.) It has sometimes been assumed in the literature that a Stratonovich Langevin equation can be quickly converted to a path integral by treating time as continuous but using the rule $\theta(t=0)=\frac{1}{2}$. I show that this prescription fails when the amplitude $e(\mathbf{q})$ is $\mathbf{q}$ dependent.

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I. INTRODUCTION

Let $\boldsymbol{\xi}$ be Gaussian white noise, which I will normalize as

$$\langle \xi_i(t)\xi_j(t')\rangle=\Omega\,\delta_{ij}(t-t'). \tag{1.1}$$

It has long been known that a Langevin equation of the form

$$\frac{d}{dt}q_i=-F_i(\mathbf{q})+\xi_i \tag{1.2}$$

can be alternatively described in terms of a path integral of the form[1]

$$\mathcal{P}(\mathbf{q}'',\mathbf{q}',t''-t')=\int_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''}[d\mathbf{q}(t)]\exp\left[-\int_{t'}^{t''}L(\dot{\mathbf{q}},\mathbf{q})\right]. \tag{1.3}$$

Here, $\mathcal{P}(\mathbf{q}'',\mathbf{q}',t)$ is the probability density that the system will end up at $\mathbf{q}''$ at time t if it started at $\mathbf{q}'$ at time zero. However, the exact form of L depends on the convention used in discretizing time when defining the path integral. With a symmetric discretization,[2]

$$\mathcal{P}(\mathbf{q}'',\mathbf{q}',t)=\lim_{\Delta t\to 0}N\int_{\mathbf{q}(0)=\mathbf{q}'}^{\mathbf{q}(t)=\mathbf{q}''}\left[\prod_t d\mathbf{q}_t\right]\times\exp\left[-\Delta t\sum_t L\left(\frac{\mathbf{q}_t-\mathbf{q}_{t-\Delta t}}{\Delta t},\frac{\mathbf{q}_t+\mathbf{q}_{t-\Delta t}}{2}\right)\right], \tag{1.4}$$

with Lagrangian

$$L(\dot{\mathbf{q}},\mathbf{q})=\frac{1}{2\Omega}|\dot{\mathbf{q}}+\mathbf{F}|^2-\frac{1}{2}F_{i,i}. \tag{1.5}$$

Here and throughout, I adopt the notation that indices after a comma represent derivatives: $F_{i,j}\equiv\partial F_i/\partial q_j$ and $F_{i,jk}\equiv\partial^2 F_i/\partial q_j\partial q_k$. N is the usual overall normalization of the path integral, which I will not bother being explicit about.

What has not been properly discussed, to my knowledge, is how to correctly form such a symmetrically discretized path integral for the case of Langevin equations with multiplicative noise [meaning noise whose amplitude $e(\mathbf{q})$ depends on $\mathbf{q}$]. Schematically,

$$\frac{d}{dt}q_i=-F_i(\mathbf{q})+e_{ia}(\mathbf{q})\xi_a, \tag{1.6}$$

with ξ as before, Eq. (1.1). I will assume that the matrix e_{ia} is invertible. There are a wide variety of applications of such equations, but I will just mention one particular example of interest to me, which motivated this work and for which a path integral formulation is particularly convenient: the calculation of the rate of electroweak baryon number violation in the early universe [3].

By itself, the continuum equation (1.6) suffers a well-known ambiguity. To define the problem more clearly, we must discretize time and take $\Delta t\to 0$. Specifically, I will interpret Eq. (1.6) in Stratonovich convention, writing

$$\mathbf{q}_t-\mathbf{q}_{t-\Delta t}=-\Delta t\,\mathbf{F}(\overline{\mathbf{q}})+e(\overline{\mathbf{q}})\boldsymbol{\xi}_t, \tag{1.7}$$

$$\langle \xi_{at}\xi_{bt'}\rangle=\Delta t\,\Omega\,\delta_{ab}\delta_{tt'}, \tag{1.8}$$

where

$$\overline{\mathbf{q}}\equiv\frac{\mathbf{q}_t+\mathbf{q}_{t-\Delta t}}{2}. \tag{1.9}$$

[The fact that I have labeled the noise $\boldsymbol{\xi}_t$ instead of $\boldsymbol{\xi}_{t-\Delta t}$ in Eq. (1.7) is just an inessential choice of convention.] The Stratonovich equation (1.7) is equivalent to the Itô equation

[1] For a review of background material in notation close to that which I use here, see, for example, Chap. 4 of Ref. [1]. The most substantial difference in notation is that my $\mathbf{F}$ is that reference's $\frac{1}{2}\mathbf{f}$.

[2] For a discussion of what changes if other discretizations are used in this case, see Ref. [2].

$$\mathbf{q}_t-\mathbf{q}_{t-\Delta t}=-\Delta t\tilde{\mathbf{F}}_i(\mathbf{q}_{t-\Delta t})+e(\mathbf{q}_{t-\Delta t})\boldsymbol{\xi}_t, \qquad (1.10)$$

with

$$\tilde{F}_i=F_i-\frac{\Omega}{2}e_{ia,j}e_{ja}. \qquad (1.11)$$

I will give two different methods for deriving the corresponding path integral. The result is

$$\mathcal{P}(\mathbf{q}'',\mathbf{q}',t)=\lim_{\Delta t\to 0}N\int_{\mathbf{q}(0)=\mathbf{q}'}^{\mathbf{q}(t)=\mathbf{q}''}\left[\prod_t d\mathbf{q}_t\right]\times\left[\prod_t \det e\left(\frac{\mathbf{q}_t+\mathbf{q}_{t-\Delta t}}{2}\right)\right]^{-1}\times\exp\left[-\Delta t\sum_t L\left(\frac{\mathbf{q}_t-\mathbf{q}_{t-\Delta t}}{\Delta t},\frac{\mathbf{q}_t+\mathbf{q}_{t-\Delta t}}{2}\right)\right], \qquad (1.12)$$

$$L(\dot{\mathbf{q}},\mathbf{q})=\frac{1}{2\Omega}(\dot{q}+F)_i g_{ij}(\dot{q}+F)_j-\frac{1}{2}F_{i,i}+\frac{1}{2}e^{-1}_{ia}e_{ka,k}(\dot{q}+F)_i+\frac{\Omega}{8}e_{ia,j}e_{ja,i}, \qquad (1.13)$$

where

$$g_{ij}\equiv(e^{-1})_{ia}(e^{-1})_{ja}. \qquad (1.14)$$

This differs from a result previously given by Zinn-Justin[3] [1] by the inclusion of the last term in L. Zinn-Justin's derivation was done in continuous time, resolving ambiguities using the prescription $\theta(t=0)=\frac{1}{2}$, which is known to work in the case where $e(\mathbf{q})$ is constant.

Because of the confusion surrounding these issues, I will show how to derive the result in two different ways. First, I will follow the standard procedure for directly turning Langevin equations into path integrals, but I will be careful to keep time discrete throughout the derivation. The second method will be to start from the Fokker-Planck equation equivalent to the Langevin equations (1.7) and (1.10) and to then turn that Fokker-Planck equation into a path integral, again using standard methods.

II. DIRECT DERIVATION FROM THE LANGEVIN EQUATION

Rewrite the discretized Langevin equation (1.7) as

$$\mathbf{E}_\tau-\boldsymbol{\xi}_\tau=0, \qquad (2.1)$$

where τ is a discrete time index and

$$\mathbf{E}_\tau\equiv e^{-1}(\overline{\mathbf{q}})[\mathbf{q}_\tau-\mathbf{q}_{\tau-1}+\Delta t\mathbf{F}(\overline{\mathbf{q}})], \qquad (2.2)$$

with $\overline{\mathbf{q}}\equiv(\mathbf{q}_\tau+\mathbf{q}_{\tau-1})/2$. The corresponding path integral is obtained by implementing these equations, for each value of τ, as δ functions, with appropriate Jacobian, integrated over the Gaussian noise distribution:

$$\mathcal{P}(\mathbf{q}'',\mathbf{q}',t)=\lim_{\Delta t\to 0}N\int_{\mathbf{q}(0)=\mathbf{q}'}^{\mathbf{q}(t)=\mathbf{q}''}\left[\prod_\tau d\boldsymbol{\xi}_\tau\exp\left(-\frac{\xi_\tau^2}{2\Omega\Delta t}\right)\times d\mathbf{q}_\tau\delta(E_\tau-\xi_\tau)\right]\det_{\tau'a;\tau''i}\left(\frac{\partial E_{\tau'a}}{\partial q_{\tau''i}}\right). \qquad (2.3)$$

The noise integral then gives

$$\mathcal{P}(\mathbf{q}'',\mathbf{q}',t)=\int\left[\prod_\tau d\mathbf{q}_\tau\right]\exp\left(-\frac{1}{2\Omega\Delta t}\sum_\tau E_\tau^2\right)\times\det_{\tau'a;\tau''i}\left(\frac{\partial E_{\tau'a}}{\partial q_{\tau''i}}\right). \qquad (2.4)$$

In our case, Eq. (2.2), the determinant takes the form

$$\det_{\tau'a;\tau''i}\left(\frac{\partial E_{\tau'a}}{\partial q_{\tau''i}}\right)=\det\begin{pmatrix}\frac{\partial\mathbf{E}_1}{\partial\mathbf{q}_1} & 0 & 0 & 0 & \\ \frac{\partial\mathbf{E}_2}{\partial\mathbf{q}_1} & \frac{\partial\mathbf{E}_2}{\partial\mathbf{q}_2} & 0 & 0 & \\ 0 & \frac{\partial\mathbf{E}_3}{\partial\mathbf{q}_2} & \frac{\partial\mathbf{E}_3}{\partial\mathbf{q}_3} & 0 & \\ 0 & 0 & \frac{\partial\mathbf{E}_4}{\partial\mathbf{q}_3} & \frac{\partial\mathbf{E}_4}{\partial\mathbf{q}_4} & \\ & & & \ddots & \ddots\end{pmatrix}=\prod_\tau\det_{ai}\left(\frac{\partial E_{\tau a}}{\partial q_{\tau i}}\right). \qquad (2.5)$$

The registration of the diagonals is determined by the nature of the initial boundary condition, which is that q_0 is fixed. From Eq. (2.2), we then have

$$\begin{aligned}\det_{\tau'a;\tau''i}\left(\frac{\partial E_{\tau'a}}{\partial q_{\tau''i}}\right)&=\prod_\tau\det_{ai}[(e^{-1})_{ai}+\tfrac{1}{2}(e^{-1})_{ak,i}\times(q_\tau-q_{\tau-1})_k+\Delta t\tfrac{1}{2}(e^{-1}\mathbf{F})_{a,i}]\\&=\left\{\prod_\tau\det[e^{-1}(\overline{\mathbf{q}}_\tau)]\right\}\times\left\{\prod_\tau\det_{ij}[\delta_{ij}+\tfrac{1}{2}e_{ja}(e^{-1})_{ak,i}\times(q_\tau-q_{\tau-1})_k+\Delta t\tfrac{1}{2}e_{ja}(e^{-1}\mathbf{F})_{a,i}]\right\},\end{aligned} \qquad (2.6)$$

where all e's and F's should now be understood as evaluated at $\overline{\mathbf{q}}_\tau$. Now rewrite the determinants in the last factor of Eq. (2.6) as exponentials in the usual way, using

[3] Specifically, Eq. (4.79) of Ref. [1]. See also Ref. [4], and Sec. 17.8 of Ref. [1], for a continuum time discussion of formulating the path integral for this problem using ghosts.

$$\det(1+A)=e^{\mathrm{tr}\ln(1+A)}=\exp\mathrm{tr}[A-\tfrac{1}{2}A^2+\cdots]. \tag{2.7}$$

To construct a path integral, we need to keep track of the terms in each time step up to and including $O(\Delta t)$, but we can ignore corrections that are higher order in Δt. For this purpose, the size of $\mathbf{q}_\tau-\mathbf{q}_{\tau-1}$ should be treated as $O(\sqrt{\Delta t})$, which is the size for which the $\dot{q}^2$ term in the action [the exponent in Eq. (1.12)] becomes $O(1)$ per degree of freedom. So, using the expansion (2.7), we get

$$\begin{aligned}\det_{ij}[&\delta_{ij}+\tfrac{1}{2}e_{ja}(e^{-1})_{ak,i}\Delta q_k+\Delta t\tfrac{1}{2}e_{ja}(e^{-1}\mathbf{F})_{a,i}]\\ &=\exp\{\tfrac{1}{2}e_{ia}(e^{-1})_{ak,i}\Delta q_k+\Delta t\tfrac{1}{2}e_{ia}(e^{-1}\mathbf{F})_{a,i}\\ &\quad-\frac{1}{8}e_{ia}(e^{-1})_{ak,m}e_{mb}(e^{-1})_{bl,i}\Delta q_k\Delta q_l+O((\Delta t)^{3/2})\}.\end{aligned} \tag{2.8}$$

It is the $\Delta q\Delta q$ term in this equation, which came from the second-order term in the expansion (2.7), that will generate the difference with the result quoted in Ref. [1]. Putting everything together, we get the path integral Eq. (1.12) with Lagrangian

$$\begin{aligned}L(\dot{\mathbf{q}},\mathbf{q})=&\frac{1}{2\Omega}(\dot{q}+F)_i g_{ij}(\dot{q}+F)_j-\frac{1}{2}F_{i,i}+\frac{1}{2}e_{ia}^{-1}e_{ka,k}\\ &\times(\dot{q}+F)_i+\frac{\Delta t}{8}e_{ia}(e^{-1})_{ak,m}e_{mb}(e^{-1})_{bl,i}\dot{q}_k\dot{q}_l.\end{aligned} \tag{2.9}$$

We can simplify this by realizing that the $\dot{q}_k\dot{q}_l$ in the last term can be replaced by its leading-order behavior in Δt. Specifically, recall that $\mathbf{q}_t-\mathbf{q}_{t-\Delta t}$ is order $\sqrt{\Delta t}$. So one can go for a large number of discrete time steps $1\ll N\ll 1/\Delta t$ without any net change in $\mathbf{q}$ at leading order in Δt. Moreover, the force $\mathbf{F}$ does not have any net effect, at leading order in Δt, over that number of steps. The result is that $\dot{q}_k\dot{q}_l$ can be replaced at leading order in Δt by its average over a large number of steps, ignoring $\mathbf{F}$ and treating the background value of $e(\overline{\mathbf{q}})$ as constant. The Gaussian integral for $\dot{\mathbf{q}}$ in (1.12) and (2.9) then gives the replacement rule

$$\dot{q}_k\dot{q}_l\rightarrow\frac{\Omega}{\Delta t}(g^{-1})_{kl} \tag{2.10}$$

at leading order in Δt. This substitution turns Eq. (2.9) into the result (1.13) presented earlier.

III. DERIVATION FROM FOKKER-PLANCK EQUATION

The Stratanovich Langevin equation (1.7) is well known to be equivalent to the Fokker-Planck equation

$$\dot{P}=\frac{\partial}{\partial q_i}\left[\frac{\Omega}{2}e_{ia}\frac{\partial}{\partial q_j}(e_{ja}P)+F_iP\right], \tag{3.1}$$

where $P=P(\mathbf{q},t)$ is the probability distribution of the system as a function of time. This is just a Euclidean Schrödinger equation, and one can transform Schrödinger equations into path integrals by standard methods. Specifically, rewrite the equation as

$$\dot{P}=-\hat{H}P, \tag{3.2}$$

with the Hamiltonian

$$\hat{H}=\frac{\Omega}{2}\hat{p}_ie_{ia}(\hat{\mathbf{q}})\hat{p}_je_{ja}(\hat{\mathbf{q}})-i\hat{\mathbf{p}}\cdot\mathbf{F}(\hat{\mathbf{q}}). \tag{3.3}$$

To obtain a path integral with symmetric time discretization, it is convenient to rewrite $\hat{H}$ in terms of Weyl-ordered operators. The Weyl order corresponding to a classical expression $\mathcal{O}(\mathbf{p},\mathbf{q},t)$ is defined as the operator $\hat{\mathcal{O}}_{\mathrm{W}}$ with

$$\langle\mathbf{q}|\hat{\mathcal{O}}_{\mathrm{W}}|\mathbf{q}'\rangle=\int_{\mathbf{p}}e^{i\mathbf{p}\cdot(\mathbf{q}'-\mathbf{q})}\mathcal{O}(\mathbf{p},\tfrac{1}{2}(\mathbf{q}+\mathbf{q}'),t). \tag{3.4}$$

For the sake of completeness, I will briefly review how to obtain Weyl ordering of operators in simple cases by considering the application of the operators to an arbitrary function $\psi(\mathbf{q})$. For example,

$$\begin{aligned}[p_iA(\mathbf{q})]_{\mathrm{W}}\psi(\mathbf{q})&=\langle\mathbf{q}|[p_iA(\mathbf{q})]_{\mathrm{W}}|\psi\rangle\\ &=\int_{\mathbf{q}'}\langle\mathbf{q}|[p_iA(\mathbf{q})]_{\mathrm{W}}|\mathbf{q}'\rangle\psi(\mathbf{q}')\\ &=\int_{\mathbf{q}'}\int_{\mathbf{p}}e^{i\mathbf{p}\cdot(\mathbf{q}-\mathbf{q}')}p_iA\left(\frac{\mathbf{q}+\mathbf{q}'}{2}\right)\psi(\mathbf{q}')\\ &=i\int_{\mathbf{q}'}\left[\frac{\partial}{\partial q_i'}\delta(\mathbf{q}-\mathbf{q}')\right]A\left(\frac{\mathbf{q}+\mathbf{q}'}{2}\right)\psi(\mathbf{q}')\\ &=-i\frac{\partial}{\partial q_i'}A\left(\frac{\mathbf{q}+\mathbf{q}'}{2}\right)\psi(\mathbf{q}')\bigg|_{\mathbf{q}'=\mathbf{q}}\\ &=-i\left\{\frac{1}{2}\left[\frac{\partial}{\partial q_i}A(\mathbf{q})\right]+A(\mathbf{q})\frac{\partial}{\partial q_i}\right\}\psi(\mathbf{q}).\end{aligned} \tag{3.5}$$

So

$$[p_iA(\mathbf{q})]_{\mathrm{W}}=\frac{1}{2}\{\hat{p}_i,A(\hat{\mathbf{q}})\}. \tag{3.6}$$

One can similarly show that

$$[p_ip_jA(\mathbf{q})]_{\mathrm{W}}=\frac{1}{4}\{\hat{p}_i,\{\hat{p}_j,A(\hat{\mathbf{q}})\}\}. \tag{3.7}$$

Now write the Hamiltonian (3.3) in terms of Weyl-ordered operators. One finds

$$\hat{\mathbf{p}}\cdot\mathbf{F}=[\mathbf{p}\cdot\mathbf{F}]_{\mathrm{W}}-\frac{i}{2}F_{i,i}, \tag{3.8}$$

$$\hat{p}_i e_{ia} \hat{p}_j e_{ja} = [\mathbf{p} g^{-1} \mathbf{p}]_{\mathrm{W}} - i[e_{ia} e_{ja,j} p_i]_{\mathrm{W}} + \frac{1}{4}(g^{-1})_{ij,ij} - \frac{1}{2}(e_{ia} e_{ja,j})_{,i}. \tag{3.9}$$

So

$$\hat{H} = [H(\mathbf{p},\mathbf{q})]_{\mathrm{W}}, \tag{3.10}$$

with

$$H(\mathbf{p},\mathbf{q}) = \frac{\Omega}{2}\mathbf{p} g^{-1}(\mathbf{q})\mathbf{p} - i\mathbf{p}_i\left[F_i(\mathbf{q}) + \frac{\Omega}{2} e_{ia}(\mathbf{q}) e_{ja,j}(\mathbf{q})\right] + u(\mathbf{q}), \tag{3.11}$$

$$u = -\frac{1}{2}F_{i,i} + \frac{\Omega}{8}(g^{-1})_{ij,ij} - \frac{\Omega}{4}(e_{ia} e_{ja,j})_{,i}. \tag{3.12}$$

The usual derivation of the path integral then gives

$$\mathcal{P}(\mathbf{q}'',\mathbf{q}',t) = \lim_{\Delta t \to 0} \int_{\mathbf{q}(0)=\mathbf{q}'}^{\mathbf{q}(t)=\mathbf{q}''} \left[\prod_\tau \frac{d\mathbf{p}_\tau d\mathbf{q}_\tau}{(2\pi)^d}\right] e^{-S(\mathbf{p},\mathbf{q})}, \tag{3.13}$$

$$S(\mathbf{p},\mathbf{q}) = \sum_\tau \left\{ -i\mathbf{p}_\tau \cdot (\mathbf{q}_\tau - \mathbf{q}_{\tau-1}) + \Delta t H\left(\mathbf{p}_\tau, \frac{\mathbf{q}_\tau + \mathbf{q}_{\tau-1}}{2}\right)\right\}. \tag{3.14}$$

Doing the **p** integrals with our Hamiltonian (3.11) then reproduces the path integral (1.4) with Lagrangian

$$L(\dot{\mathbf{q}},\mathbf{q}) = \frac{1}{2\Omega}(\dot{q}_i + F_i + \tfrac{1}{2}\Omega e_{ia} e_{ka,k}) g_{ij} (\dot{q}_j + F_j + \tfrac{1}{2}\Omega e_{jb} e_{lb,l}) + u(\mathbf{q}). \tag{3.15}$$

Now note that

$$(g^{-1})_{ij,ij} = 2 e_{ia} e_{ja,ij} + e_{ia,i} e_{ja,j} + e_{ia,j} e_{ja,i}, \tag{3.16}$$

and so

$$\begin{aligned}\frac{\Omega}{8}(e_{ia} e_{ka,k}) g_{ij} (e_{jb} e_{lb,l}) + u &= \frac{\Omega}{8} e_{ia,i} e_{ja,j} + \left[-\frac{1}{2}F_{i,i} + \frac{\Omega}{8}(g^{-1})_{ij,ij} - \frac{\Omega}{4}(e_{ia} e_{ja,j})_{,i}\right] \\ &= -\frac{1}{2}F_{i,i} + \frac{\Omega}{8} e_{ia,j} e_{ja,i}.\end{aligned} \tag{3.17}$$

Combining Eqs. (3.15) and (3.17) reproduces the Lagrangian (1.13) asserted in the introduction.

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